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Journal of Complexity 19 (2003) 638–643

Journal of
COMPLEXITY

<http://www.elsevier.com/locate/jco>

Multivariate integration in $C^\infty([0, 1]^d)$ is not strongly tractable

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Received 9 June 2003; accepted 26 June 2003

Abstract

It has long been known that the multivariate integration problem for the unit ball in $C^r([0, 1]^d)$ is intractable for fixed finite r . H. Woźniakowski has recently conjectured that this is true even if $r = \infty$. This paper establishes a partial result in this direction. We prove that multivariate integration for infinitely differential functions is not strongly tractable.

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1. Introduction

Multivariate integration is a classical problem of numerical analysis that has been studied for various normed spaces F_d of functions of d variables. In practical applications d is frequently very large, even in the hundreds or thousands.

Tractability and strong tractability of multivariate integration has been recently thoroughly analyzed. These concepts are defined as follows. We consider the worst case setting and define $n(\varepsilon, B_d)$ as the minimal number of function values that are needed to approximate the integral of any f from the unit ball B_d of F_d with an error threshold of ε . We want to know how $n(\varepsilon, B_d)$ depends on d and ε^{-1} . The problem is *tractable* if $n(\varepsilon, B_d)$ is bounded by a polynomial in ε^{-1} and d , and *strongly tractable* if the bound depends only polynomially on ε^{-1} , with no dependence on d .

The dependence of $n(\varepsilon, B_d)$ on ε^{-1} has been studied for many years, and bounds for $n(\varepsilon, B_d)$ in terms of ε^{-1} are known for many B_d . For instance, let $F_d = C^r([0, 1]^d)$ be the space of r times continuously differentiable functions defined on the

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d -dimensional unit cube with the norm given as the supremum of the absolute values of all partial derivatives up to the r th order. Bakhvalov proved in 1959, see [NW01], that there exist two positive numbers $C_{r,d}$ and $C_{r,d}$ such that

$$c_{r,d}\varepsilon^{-d/r} \leq n(\varepsilon, B_d) \leq C_{r,d}\varepsilon^{-d/r} \quad \forall \varepsilon \in (0, 1).$$

When r is fixed this implies that multivariate integration in $C^r([0, 1]^d)$ is intractable. Indeed, $n(\varepsilon, B_d)$ cannot possibly be bounded by a polynomial in ε^{-1} and d since the exponent of ε^{-1} goes to infinity with d , and $n(\varepsilon, B_d)$ is exponential in d . However if r varies, that is, when we consider the spaces $F_d = C^{r(d)}([0, 1]^d)$ with $\sup_d d/r(d) < \infty$, the behavior of $n(\varepsilon, B_d)$ is not known since we do not have sharp bounds on $c_{r,d}$ and $C_{r,d}$. In fact, the best known bounds on $c_{r,d}$ are exponentially small in d , while the best known bounds on $C_{r,d}$ are exponentially large in d , see again [NW01]. Thus, we can neither claim nor deny tractability or strong tractability in the class $C^{r(d)}([0, 1]^d)$ with $\sup_d d/r(d) < \infty$ on the basis of Bakhvalov's result.

The conjecture formulated in [W03] states that multivariate integration in $C^{r(d)}([0, 1]^d)$ is intractable even if $r(d) = \infty$. That is, even when we consider infinitely differentiable functions with all partial derivatives bounded by one, $n(\varepsilon, B_d)$ cannot be bounded by a polynomial in ε^{-1} and d . Although we are not able to establish this conjecture in full generality, we shall prove that multivariate integration in $C^\infty([0, 1]^d)$ is not *strongly* tractable. This is achieved by showing that for a fixed n , the n th minimal error goes to one as d approaches infinity. More precisely, we show that for any n and η there exists $d = d(n, \eta)$ such that for any linear algorithm there exists a polynomial that is a sum of univariate polynomials, which belongs to the unit ball of $C^\infty([0, 1]^d)$, whose integral is at least $1 - \eta$, and the algorithm outputs zero. This proof technique allows us to prove the lack of strong tractability, but seems too weak to establish the lack of tractability.

2. Multivariate integration in $C^\infty([0, 1]^d)$

We precisely define the problem of multivariate integration studied in this paper. Let $F_d = C^\infty([0, 1]^d)$ be the space of real functions defined on the unit cube $[0, 1]^d$ that are infinitely differentiable with the norm

$$\|f\|_d = \sup\{|D^\alpha f(x)| : x \in [0, 1]^d, \alpha - \text{any multi-index}\}.$$

Here $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_d]$ with non-negative integers α_j , $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_d$, and

$$D^\alpha f = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_d^{\alpha_d}} f$$

stands for the differentiation operator. Let B_d denote the unit ball of this space. The multivariate integration problem is defined as an approximation of integrals

$$I_d(f) = \int_{[0,1]^d} f(t) dt \quad \forall f \in B_d,$$

where the integral is taken with respect to the Lebesgue measure.

It is well known that adaption and the use of non-linear algorithms do not help for the multivariate integration problem, as proven in [B71], see also [NW01]. That is why we consider only linear algorithms,

$$A_{n,d}(f) = \sum_{j=1}^n a_j f(x_j)$$

for some real coefficients a_j and some sample points x_j from $[0, 1]^d$. Here n denotes the number of function values used by the algorithm. Of course the a_i and x_i may depend on n . The (worst case) error of the algorithm $A_{n,d}$ is defined as

$$\text{err}(A_{n,d}) = \sup\{|I_d(f) - A_{n,d}(f)| : f \in B_d\}.$$

Let $\text{LIN}_{n,d}$ denote the class of all linear algorithms that use n function values. The n th minimal error is defined as

$$e(n, B_d) = \inf\{\text{err}(A_{n,d}) : A_{n,d} \in \text{LIN}_{n,d}\}.$$

We shall prove the following theorem:

Theorem 2.1. *For any positive integer n we have*

$$\lim_{d \rightarrow \infty} e(n, C^\infty([0, 1]^d)) = 1.$$

This theorem easily implies that multivariate integration in $C^\infty([0, 1]^d)$ is *not* strongly tractable. Indeed, were it strongly tractable, we would have a polynomial bound on $n(\varepsilon, B_d)$ independent of d , thus having a linear algorithm of error at most ε and using at most $n = n(\varepsilon, B_d)$ function values. Taking, say, $\varepsilon < \frac{1}{2}$ and $n > n(\frac{1}{2}, B_d)$ we would get $e(n, B_d) \leq \frac{1}{2}$ independently of d , and this would contradict the theorem that $e(n, B_d)$ goes to one when d approaches infinity.

2.1. Proof of the theorem

We take an arbitrary positive integer n and $\eta \in (0, 1)$. The idea of the proof is to separate variables and, for sufficiently large d and any $A_{n,d} \in \text{LIN}_{n,d}$, to find a polynomial $f \in B_d$ that is a sum of univariate polynomials such that $|I_d(f) - A_{n,d}(f)| > 1 - \eta$. It will suffice to find such a polynomial for which $f(x_j) = 0$ at all points x_j used by $A_{n,d}$. Then $A_{n,d}(f) = 0$, and thus $\text{err}(A_{n,d}) \geq |I_d(f)| > 1 - \eta$. Since

$A_{n,d}$ is an arbitrary linear algorithm this implies that $e(n, B_d) \geq 1 - \eta$ for sufficiently large d . For the zero algorithm $A_{n,d} \equiv 0$, we have $\text{err}(A_{n,d}) = 1$ and hence we have $e(n, B_d) \leq 1$. Since η can be arbitrarily small, this completes the proof of Theorem 2.1.

Suppose for the moment that we have the following lemma, which will be proven in the next subsection.

Lemma 2.2. *For any positive integer n , and any $\eta \in (0, 1)$ there exists a constant $K_{\eta,n}$ such that for any choice of $y_1, y_2, \dots, y_n \in [0, 1]$ there exists a polynomial $f: [0, 1] \rightarrow \mathbb{R}$ satisfying the following conditions:*

1. $\max_{x \in [0,1]} |f(x)| \leq 1$,
2. $\max_{k=0,1,\dots} \max_{x \in [0,1]} |f^{(k)}(x)| \leq K_{\eta,n}$,
3. $\int_0^1 f(x) dx > 1 - \eta$,
4. $f(y_j) = 0$ for $j = 1, 2, \dots, n$.

From Lemma 2.2, we take any $d \geq K_{\eta,n}$ and any $A \in \text{LIN}_{n,d}$ that uses sample points

$$x_j = [x_j^1, x_j^2, \dots, x_j^d] \in [0, 1]^d$$

for $j = 1, 2, \dots, n$.

For $i = 1, 2, \dots, d$, let f_i be the polynomial given by Lemma 2.2 for $y_j = x_j^i$, with $j = 1, 2, \dots, n$. Consider the multivariate polynomial

$$f(t_1, t_2, \dots, t_d) = \frac{1}{d} \sum_{i=1}^d f_i(t_i), \quad t_i \in [0, 1].$$

The values of f are bounded by 1 since they are arithmetic means of the values of f_i from $[-1, 1]$. Any mixed derivative of such a function f is 0, while

$$\left| \frac{\partial^k}{\partial x_i^k} f(a) \right| = \left| \frac{1}{d} f_i^{(k)}(a^i) \right| \leq \frac{K_{\eta,n}}{d} \leq 1.$$

Thus f belongs to B_d . Additionally,

$$\int_{[0,1]^d} f(t) dt = \frac{1}{d} \sum_{i=1}^d \int_{[0,1]} f_i(x) dx > 1 - \eta.$$

Furthermore $f(x_j) = 0$ since $f_i(x_j^i) = 0$ for all $j = 1, 2, \dots, n$. Thus f is a function we needed to prove Theorem 2.1.

2.2. Proof of the Lemma

We will use the Stone–Weierstrass theorem to find a function satisfying Lemma 2.2.

For $\delta = \eta/(7n)$, let the function $g: [0, 2 + \delta] \rightarrow \mathbb{R}$ be defined as $1 - \delta$ on $[0, 1 - \delta] \cup [1 + \delta, 2 + \delta]$, -2δ at 1 and linear on $[1 - \delta, 1]$ and $[1, 1 + \delta]$. It is obviously a continuous function, so by the Stone–Weierstrass theorem we can approximate it by

a polynomial P of degree $N = N(\eta, n)$ such that

$$\max_{x \in [0, 2+\delta]} |g(x) - P(x)| < \delta.$$

The polynomial P is negative at 1 and positive at $1 + \delta$, so it has a root at some $y_0 \in (1, 1 + \delta)$. Let $P_i(x) = P(x + y_0 - y_i)$. As $y_0 - y_i \in (0, 1 + \delta)$ the polynomial P_i satisfies $\max_{x \in [0, 1]} |P_i(x) - g(x + y_0 - y_i)| < \delta$. Now take

$$f(x) = \prod_{i=1}^n P_i^2(x) \quad \forall x \in [0, 1].$$

Note that $f(y_j) = \prod_{i=1}^n P_i^2(y_j + y_0 - y_i) = 0$ since the j th factor is $P^2(y_0) = 0$.

The polynomial P_i satisfies

$$1 - 2\delta < P_i(x) < 1 \quad \forall x \in [0, 1] \setminus [y_i - 2\delta, y_i + \delta].$$

Thus,

$$(1 - 2\delta)^{2n} < f(x) < 1 \quad \forall x \in [0, 1] \setminus \bigcup_{i=1}^n [y_i - 2\delta, y_i + \delta]$$

and, of course, $f(x) \geq 0$ on the whole interval $[0, 1]$. This allows us to approximate the integral of f . Indeed,

$$\int_0^1 f(x) dx \geq \int_{[0, 1] \setminus \bigcup_{i=1}^n [y_i - 2\delta, y_i + \delta]} f(x) dx > (1 - 3\delta n)(1 - 2\delta)^{2n}.$$

Using the Bernoulli inequality we conclude that the last expression is at least

$$(1 - 3n\delta)(1 - 4n\delta) \geq 1 - 7n\delta = 1 - \eta.$$

The function f is a polynomial on $[0, 1]$. Its coefficients are continuous functions of $(y_1, y_2, \dots, y_n) \in [0, 1]^n$ since the coefficients of each P_i are continuous functions of y_i , and f is the product of P_i 's. The upper bound of the j th derivative of f is a continuous function of f 's coefficients, and thus a continuous function of (y_1, y_2, \dots, y_n) . As a continuous function on a compact set it is bounded for each j , and so all derivatives up to the $2nN$ th order have a common bound, say, $K_{\eta, n}$, independent of (y_1, y_2, \dots, y_n) . This means that the second condition of Lemma 2.2 is satisfied and the proof of Lemma 2.2 is completed.

Acknowledgments

This work was completed with the enormous help of Prof. Henryk Woźniakowski, who suggested the problem, gave me the theoretical tools to solve it and worked with me on creating a publishable paper from the mathematical reasoning. The help and support of my father was also crucial to finish this work.

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